Nonlinear Hyperbolic Equations with Dissipative Temporal and Spatial Non-Local Memory

František Mošna, Jindřich Nečas

The equation governing the evolution of a displacement vector in an elastic body with dissipative temporal and spatial non-local memory is considered. The memory term is generated by a singular but integrable kernel. The existence of a global weak solution to an associated initial-boundary problem is established by constructing Galerkin approximations and deriving suitable energy estimate.

Key words: *singular viscoelasticity, energy estimates, weak solution*

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1. Introduction.

In this paper, the equation governing the evolution of a displacement vector in an elastic body is investigated.

The body is assumed to occupy a reference configuration $\Omega \subset \mathbb{R}^N$ at an initial time and to have unit density. The vector $u = (u_1, \dots, u_N)$ represents the displacement. Our investigation is focused on the existence of a weak solution to the non-linear wave equation for the displacement u written in the form

$$\frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j}\sigma_{ij} = f_i \qquad i = 1, \dots, N, \tag{1}$$

where σ_{ij} is the Cauchy stress tensor and $f = (f_1, \dots, f_N)$ is the external body force per unit mass.

The stress tensor usually depends on the symmetrized gradient of the displacement vector (infinitesimal strain tensor) given by the constitutive equation

$$\sigma_{ij}(x,t) = \frac{\partial W}{\partial e_{ij}}(eu(x,t)), \tag{2}$$

where $W = W(e_{ij})$ is the function of free energy and $e_{ij}u = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. In the case of one space dimension it is well known that a weak solution exists. This result follows from the recent work of Di Perna [6] and is based on compensated compactness arguments. Intensive efforts of many mathematicians have brought partial results in some special cases for dimension $N \geq 2$. However, the question of existence of a solution to the general nonlinear elastic problem remains open.

Experience indicates that certain materials have memory. It means that the stress depends not only on the strain at the present time t, but also on the entire history of the strain from zero to time t. In this case, the instantaneous stress (2) in equation (1) is extended by the memory part, which usually has the form

$$-\int_0^t h(\tau - t)(eu(x,\tau) - eu(x,t))\, d\tau, \tag{3}$$

(h denotes a suitable kernel). On first sight it is surprising that existence of a solution to such an equation can be proved (see [2] and [14]).

However, there are other materials where the stress depends not only on the history of the strain at given x, but also on the history at all points located in the neigbourhood of x, more generally on the history at all points of Ω. To prove the existence of a weak solution for the system including non/local memory effects, both time and spatial ones, is the purpose of this paper.

We will consider the equation

$$\ddot{u}_i(x,t) - \frac{\partial}{\partial x_j}\sigma_{ij}(x,t) = f_i(x,t) \qquad \text{on } \Omega \times (0,\infty), \qquad i = 1,2,\ldots,N \tag{4}$$

where $\sigma = \sigma^I + \sigma^M$,

$$\sigma^I_{ij} = \frac{\partial W}{\partial e_{ij}}(eu) \quad , \tag{5}$$

$$\sigma^M_{ij} = -\lambda \int_0^t \int_\Omega (e_{ij}u(\xi,\tau) - e_{ij}u(\xi,t)) \frac{h(t-\tau)}{|x-\xi|^\alpha} d\xi \; d\tau \tag{6}$$

with boundary conditions

$$u(x,\cdot) = 0 \qquad \text{for} \qquad x \in \partial\Omega \tag{7}$$

and initial conditions

$$u(\cdot,0) = u^0 \qquad \dot{u}(\cdot,0) = u^1. \tag{8}$$

Let a domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be bounded and let it posses Lipschitz continuous boundary $\partial\Omega$. We assume that the function $W : \mathbb{R}^{2N} \to \mathbb{R}$ is continuous, has bounded second derivatives, $W(0) = \frac{\partial W}{\partial e_{ij}}(0) = 0$ and the condition of ellipticity holds, e.g. there exists a real number $\kappa > 0$ such that

$$\frac{\partial^2 W}{\partial e_{ij}\,\partial e_{kl}}(q) \cdot a_{ij} \cdot a_{kl} \geq \kappa \cdot ||a||^2 \tag{9}$$

holds for every $a,\ q \in \mathbb{R}^{2N}$.

We also suppose $h(t) = e^{-t}t^{-\nu}$, where $0 < \nu < \frac{1}{2}$, $N-1 < \alpha < N$, $\lambda > 0$ and

$$f \in W^{\frac{\nu}{2},2}((0,\infty); W^{1-\frac{N-\alpha}{2},2}(\Omega;\mathbb{R}^N)) \cap L^2((0,\infty); W^{-1,2}(\Omega;\mathbb{R}^N)) \cap L^\infty((0,\infty); L^2(\Omega;\mathbb{R}^N)), \tag{10}$$

$$u^0 \in W^{1,2}_o(\Omega;\mathbb{R}^N), \qquad u^1 \in L^2(\Omega;\mathbb{R}^N). \tag{11}$$

We use the Galerkin approximation. The operator $\frac{\partial}{\partial x_j}\sigma_{ij}$ is compact in time and space. The memory part of the stress tenzor allows us to establish the basic estimates.

We will deal with spaces of functions with non-integer derivatives.

Definition 1. Let $0 \leq s < 2$ and $u : \Omega \to B$ be a function, where B is a Banach space and $\Omega \subset \mathbb{R}^N$ is a domain with Lipschitz continuous boundary. We define

$$||u||^2_{W^{s,2}(\Omega;B)} = ||u||^2_{L^2(\Omega;B)} + \int_\Omega \int_\Omega \frac{||u(x)-u(y)||^2_B}{|x-y|^{N+2s}}\, dx\; dy \qquad \text{for } 0 < s < 1, \tag{12}$$

and

$$\|u\|^2_{W^{s,2}(\Omega;B)} = \|u\|^2_{W^{1,2}(\Omega;B)} + \sum_{i=1}^{N} \int_\Omega \int_\Omega \frac{\|\frac{\partial u}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(y)\|^2_B}{|x-y|^{N+2(s-1)}}\, dx\, dy \quad \text{for } 1 < s < 2. \tag{13}$$

The space $W^{s,2}(\Omega; B)$ contains the functions u satisfying

$$\|u\|_{W^{s,2}(\Omega;B)} < \infty, \tag{14}$$

$W^{0,2}(\Omega; B) = L^2(\Omega; B)$, $W^{1,2}(\Omega; B)$ is introduced usually. (If $B = \mathbb{R}$, then we denote $W^{s,2}(\Omega) = W^{s,2}(\Omega; \mathbb{R})$.)

The space $W^{s,2}_o(\Omega)$ can be introduced as the closure of $\mathcal{D}(\Omega)$ (test functions) in $W^{s,2}(\Omega)$ and we denote the dual space $W^{-1,2}(\Omega) = \left(W^{1,2}_o(\Omega)\right)^*$.

For $-\frac{1}{2} < s < \frac{1}{2}$ we have $W^{s,2}(\Omega) = W^{s,2}_o(\Omega)$.

Let $v^1, v^2, \ldots$ be a basis in $W^{1,2}_o(\Omega)$ which is orthonormal in $L^2(\Omega)$ composed by the eigenfunctions of the equation

$$\Delta v + \lambda v = 0 \qquad \text{on } \Omega, \qquad v \in W^{1,2}_o(\Omega) \tag{15}$$

and we denote the corresponding eigenvalues $\lambda_1, \lambda_2, \ldots$. In the space $W^{s,2}_o(\Omega)$ there exists an equivalent norm

$$\|u\|^2_{W^{s,2}_o(\Omega)} \approx \sum_{i=1}^{\infty} \lambda_i^s \cdot c_i^2, \quad \text{where} \quad c_i = \int_\Omega u v^i\, dx \tag{16}$$

and

$$\|u\|^2_{W^{s,2}_o(\mathbb{R}^N)} \approx \int_{\mathbb{R}^N} (|\xi|^s \cdot |\widehat{u}(\xi)|)^2\, d\xi \tag{17}$$

where $\widehat{u}$ means Fourier transform of function u

$$\widehat{u}(\xi) = \int_{\mathbb{R}^N} u(x) \cdot e^{-i\cdot(\xi_1 x_1 + \cdots + \xi_N x_N)}\, dx \quad , \qquad \xi \in \mathbb{R}^N. \tag{18}$$

We shall use the Parseval equality

$$\int_{\mathbb{R}^N} u \cdot \bar{v}\, dx = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \widehat{u} \cdot \bar{\widehat{v}}\, d\xi \qquad u, v \in L^2(\mathbb{R}^N), \tag{19}$$

rules for Fourier transform of convolution and derivatives

$$\widehat{u * v} = \widehat{u} \cdot \widehat{v}, \tag{20}$$

$$\widehat{\frac{\partial u}{\partial x_j}}(\xi) = -i \cdot \xi_j \cdot \widehat{u}(\xi), \quad u \in \mathcal{S}^*(\mathbb{R}^N),\ v \in L^2(\mathbb{R}^N). \tag{21}$$

(The space of temperate distributions $\mathcal{S}^*(\mathbb{R}^N)$ means the dual space to

$$\mathcal{S}(\mathbb{R}^N) = \{\varphi \in C^\infty(\mathbb{R}^N)\ ;\ \sup_{x\in\mathbb{R}^N} |x^\beta D^\alpha \varphi(x)| < \infty \ \text{ for every multiindex } \alpha, \beta \in \mathbb{N}^N\} \tag{22}$$

and convolution of u and v is introduced by the formula

$$(u*v)(x)=\int_{\mathbb{R}^N} u(\xi)v(x-\xi)\,d\xi.) \tag{23}$$

We shall need the Fourier transformation of power $\frac{1}{|\cdot|^\alpha}$ for $\frac{N-1}{2}<\alpha<N$

$$\widehat{\left(\frac{1}{|\cdot|^\alpha}\right)}(\xi)=(2\pi)^{\frac{N}{2}}\cdot 2^{\frac{N}{2}-\alpha}\cdot\frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}\cdot\left(\sqrt{\xi_1^2+\cdots+\xi_N^2}\right)^{\alpha-N}. \tag{24}$$

2. Galerkin approximation.

Definition 2. A weak solution to a mixed problem (4) — (8) is a function $u\in L^\infty((0,\infty);W_o^{1,2}(\Omega;\mathbb{R}^N))$, for which

$$\begin{aligned}&\dot u\in L^\infty((0,\infty);L^2(\Omega;\mathbb{R}^N))\\ &\ddot u\in L^2((0,T);W^{-1,2}(\Omega;\mathbb{R}^N))\qquad\text{for all } T>0\end{aligned} \tag{25}$$

and for all $v\in W_o^{1,2}(\Omega;\mathbb{R}^N)$, and for almost all $T>0$ the equality

$$\begin{aligned}&\int_0^T\int_\Omega \ddot u_i(x,t)\,v_i(x)\,dxdt+\int_0^T\int_\Omega\frac{\partial W}{\partial e_{ij}}(eu(x,t))\cdot e_{ij}v(x)\,dxdt-\\ &-\lambda\int_0^T\int_\Omega\left(\int_0^t\int_\Omega(e_{ij}u(\xi,\tau)-e_{ij}u(\xi,t))\cdot\frac{h(t-\tau)}{|x-\xi|^\alpha}\cdot d\xi\;d\tau\right)\cdot e_{ij}v(x)\,dxdt=\\ &=\int_0^T\int_\Omega f_i(x,t)\cdot v_i(x)\,dxdt\end{aligned} \tag{26}$$

holds.

(It is necessary to comprehend the integrals in the sense of distributions.)

There exists a basis $w^1,w^2,\ldots$ in the space $W_o^{1,2}(\Omega)$ which is orthonormal in $L^2(\Omega)$. We construct Galerkin approximants u^n of the form

$$u^n(x,t)=\sum_{k=1}^n c_k^{(n)}(t)\cdot w^k(x),\qquad n=1,2,\ldots. \tag{27}$$

Using successively $w^1,\ldots,w^n$ as test functions in (26), we get conditions for functions of time $c_1^{(n)},c_2^{(n)},\ldots,c_n^{(n)}$

$$\begin{aligned}&\ddot c_m^{(n)}(t)+\int_\Omega\frac{\partial W}{\partial e_{ij}}\left(\sum_{k=1}^n c_k^{(n)}(t)\cdot ew^k(x)\right)\cdot e_{ij}w^m(x)\,dx-\\ &-\lambda\sum_{k=1}^n\int_\Omega\int_\Omega\frac{e_{ij}w^k(\xi)e_{ij}w^m(x)d\xi\,dx}{|x-\xi|^\alpha}\cdot\int_0^t\left(c_k^{(n)}(\tau)-c_k^{(n)}(t)\right)\cdot h(t-\tau)\,d\tau=\\ &=\int_\Omega f_i(x,t)\cdot w_i^m(x)\,dx\end{aligned} \tag{28}$$

with conditions

$$c_m^{(n)}(0)=\int_\Omega u_i^0\cdot w_i^m\,dx,\qquad \dot c_m^{(n)}(0)=\int_\Omega u_i^1\cdot w_i^m\,dx,\qquad m=1,\ldots,n. \tag{29}$$

This problem possesses a unique solution on some interval $\langle 0, T_\infty)$, (T_∞ is the maximal time of existence of the solution).

Thus there exist approximate solutions u^n satisfying the equation

$$
(30)\quad \int_\Omega \ddot{u}_i^n\, v_i\, dx + \int_\Omega \frac{\partial W}{\partial e_{ij}}(eu^n)\cdot e_{ij} v\, dx -
$$
$$
- \lambda \int_\Omega \left(\int_0^t \int_\Omega (e_{ij}u^n(\xi,\tau) - e_{ij}u^n(\xi,t)) \cdot \frac{h(t-\tau)}{|x-\xi|^\alpha} \cdot d\xi\; d\tau \right) \cdot e_{ij} v(x)\, dx =
$$
$$
= \int_\Omega f_i \cdot v_i\, dx
$$

for all $v \in sp\{w^1, \ldots, w^n\}$ (subspace spanned by $w^1, \ldots, w^n$).

3. Basic estimates.

For this solution we may state the following.

Lemma 1. For any T, $0 \le T < T_\infty$ the solution u^n satisfies for some $C_1 > 0$

$$
(31)\quad \frac{1}{2}\int_\Omega \|\dot{u}^n(\cdot,T)\|^2\, dx + \int_\Omega W(eu^n(\cdot,T))\, dx +
$$
$$
+ \lambda C_1 \int_0^T \int_0^T \int_{\mathbb{R}^N} \left(|\xi|^{1-\frac{N-\alpha}{2}} |\widehat{u}(\xi,t) - \widehat{u}(\xi,\tau)| \right)^2 d\xi \frac{d\tau dt}{(t-\tau)^{1+\nu}} \le
$$
$$
\le \frac{1}{2}\int_\Omega \|u^1\|^2\, dx + \int_\Omega W(eu^0)\, dx + \int_0^T \int_\Omega f_i \dot{u}_i^n\, dx dt.
$$

Proof: Let us extend u^n by zero outside Ω. We put the time derivatives $\dot{u}^n(\cdot,t)$ as test functions into the expression (30) and integrate over $(0,T)$

$$
(32)\quad \frac{1}{2}\int_\Omega \|\dot{u}^n(\cdot,T)\|^2\, dx - \frac{1}{2}\int_\Omega \|\dot{u}^n(\cdot,0)\|^2\, dx +
$$
$$
+ \int_\Omega W(eu^n(\cdot,T))\, dx - \int_\Omega W(eu^n(\cdot,0))\, dx +
$$
$$
+ \lambda \int_0^T \int_\Omega \left(\int_0^t \int_\Omega (e_{ij}u^n(\xi,t) - e_{ij}u^n(\xi,\tau)) \cdot \frac{h(t-\tau)}{|x-\xi|^\alpha}\, d\xi d\tau \right) \dot{e}_{ij} u^n(x,t)\, dx dt =
$$
$$
= \int_0^T \int_\Omega f_i \cdot \dot{u}_i^n\, dx dt.
$$

We can write the last integral on the left-hand side of (32) as a convolution (23), then we use the Parseval equality (19) and properties of Fourier transform of convolution, derivatives and powers (20), (21), (24)

$$
(33)\quad \int_\Omega \int_\Omega \frac{e_{ij}u^n(\xi,t) - e_{ij}u^n(\xi,\tau)}{|x-\xi|^\alpha} e_{ij}\dot{u}^n(x,t)\, d\xi dx =
$$
$$
= \int_{\mathbb{R}^N} \left[(e_{ij}u^n(\cdot,t) - e_{ij}u^n(\cdot,\tau)) * \frac{1}{|\cdot|^\alpha} \right](x) \cdot \overline{e_{ij}\dot{u}^n(x,t)}\, dx =
$$
$$
= (2\pi)^{\frac{N}{2}} \cdot 2^{\frac{N}{2}-\alpha} \cdot \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \cdot \frac{1}{2} \left[\int_{\mathbb{R}^N} |\xi|^{2+\alpha-N} \left(\widehat{u_i^n}(\xi,t) - \widehat{u_i^n}(\xi,\tau) \right) \cdot \overline{\widehat{\dot{u}_i^n}(\xi,t)}\, d\xi + \right.
$$
$$
\left. + \int_{\mathbb{R}^N} |\xi|^{\alpha-N} \xi_i \xi_j \left(\widehat{u_i^n}(\xi,t) - \widehat{u_i^n}(\xi,\tau) \right) \cdot \overline{\widehat{\dot{u}_j^n}(\xi,t)}\, d\xi \right].
$$

Denoting by $\mathbf{n}=(n_t, n_\tau)$ the outer normal to ∂M_T, where

$$M_T = \{(t,\tau); 0 < t < \tau, 0 < \tau < T\},$$

we compute that for some $d_1 > 0$

$$\begin{aligned}(34)\quad &\int_0^T \int_0^t \int_{\mathbb{R}^N} |\xi|^{2+\alpha-N} \left(\widehat{u_i^n}(\xi,t) - \widehat{u_i^n}(\xi,\tau)\right) \cdot \overline{\widehat{u_i^n}(\xi,t)} d\xi \; h(t-\tau)\, d\tau dt = \\ &= \int_{\mathbb{R}^N} |\xi|^{2+\alpha-N} \int_{M_T} \frac{d}{dt} \left|\widehat{u^n}(\xi,t) - \widehat{u^n}(\xi,\tau)\right|^2 \cdot h(t-\tau)\, d\tau dt d\xi = \\ &= \int_{\mathbb{R}^N} |\xi|^{2+\alpha-N} \left(\int_{\partial M_T} \left|\widehat{u^n}(\xi,t) - \widehat{u^n}(\xi,\tau)\right|^2 \cdot h(t-\tau) n_t \, dS - \right. \\ &\left. - \int_{M_T} \frac{d}{dt} \left|\widehat{u^n}(\xi,t) - \widehat{u^n}(\xi,\tau)\right|^2 \cdot h'(t-\tau)\, d\tau dt \right) d\xi \geq \\ &\geq d_1 \int_0^T \int_0^T \int_{\mathbb{R}^N} \left(|\xi|^{1-\frac{N-\alpha}{2}} |\widehat{u^n}(\xi,t) - \widehat{u^n}(\xi,\tau)| \right)^2 d\xi \, \frac{d\tau dt}{(t-\tau)^{1+\nu}},\end{aligned}$$

(we use integration by parts and consider that h' is negative on $(0,\infty)$).

Similarly

$$\begin{aligned}(35)\quad &\int_0^T \int_0^t \int_{\mathbb{R}^N} |\xi|^{\alpha-N} \xi_i \xi_j \left(\widehat{u_i^n}(\xi,t) - \widehat{u_i^n}(\xi,\tau)\right) \cdot \overline{\widehat{u_j^n}(\xi,t)} \cdot h(t-\tau)\, d\tau dt = \\ &= \int_{\mathbb{R}^N} |\xi|^{\alpha-N} \int_{M_T} \frac{d}{dt} \left[\sum_{i=1}^N \xi_i \left(\widehat{u_i^n}(\xi,t) - \widehat{u_i^n}(\xi,\tau) \right) \right]^2 \cdot h(t-\tau)\, d\tau dt d\xi \geq 0. \blacksquare\end{aligned}$$

Lemma 2. For any $T > 0$ there exists a constant $C_2(T) > 0$ such that the Galerkin approximants u^n satisfy

$$(36)\qquad \|\ddot{u}^n\|_{W^{\frac{\nu}{2},2}((0;T);W^{-1-\frac{N-\alpha}{2},2}(\Omega;\mathbb{R}^N))} \leq C_2(T).$$

Proof:

We denote $\epsilon = \frac{N-\alpha}{2}$ and let R^n be the projection operator mapping $W_o^{1+\epsilon,2}(\Omega;\mathbb{R}^N)$ to $sp(w^1,\ldots,w^n)$.

The starting point of our consideration will be the definition

$$(37)\qquad \|\ddot{u}^n\|^2_{W^{\frac{\nu}{2},2}((0;T);W^{-1-\epsilon,2}(\Omega;\mathbb{R}^N))} \approx \int_0^T \int_0^T \frac{\|\ddot{u}^n(\cdot,t_1) - \ddot{u}^n(\cdot,t_2)\|^2_{W^{-1-\epsilon,2}}}{|t_1-t_2|^{1+\nu}} \, dt_1 dt_2,$$

(38)

$$\begin{aligned}\|\ddot{u}^n(\cdot,t_1) - \ddot{u}^n(\cdot,t_2)\|_{W^{-1-\epsilon,2}(\Omega;\mathbb{R}^N)} &= \sup_{\|\psi\|_{W_0^{1+\epsilon,2}} \leq 1} \int_\Omega \left(\ddot{u}_i^n(\cdot,t_1) - \ddot{u}_i^n(\cdot,t_2)\right) \cdot \psi_i \, dx \\ &= \sup_{\|\psi\|_{W_0^{1+\epsilon,2}} \leq 1} \int_\Omega \left(\ddot{u}_i^n(\cdot,t_1) - \ddot{u}_i^n(\cdot,t_2)\right) \cdot (R^n\psi)_i \, dx.\end{aligned}$$

If we extend any function $\varphi \in W_o^{1+\epsilon,2}(\Omega;\mathbb{R}^N)$ by zero outside Ω then

$$\|\varphi\|^2_{W_o^{s,2}(\mathbb{R}^N)} \approx \int_{\mathbb{R}^N} \left(|\xi|^s \cdot |\widehat{\varphi}(\xi)|\right)^2 \, d\xi, \qquad -\frac{3}{2} < s < \frac{3}{2}. \tag{39}$$

We choose any function $\psi \in W_o^{1+\epsilon,2}(\Omega;\mathbb{R}^N)$, $\|\psi\|_{W_o^{1+\epsilon,2}(\Omega;\mathbb{R}^N)} \le 1$, denote $\varphi = R^n\psi \in sp(w^1,\dots,w^n)$ and use equality (20) for $u^n(\cdot,t_1)$ and $u^n(\cdot,t_2)$.

First we estimate

$$\begin{aligned}
&\left|\int_\Omega \left(\frac{\partial W}{\partial e_{ij}}\,(eu^n(x,t_1)) - \frac{\partial W}{\partial e_{ij}}\,(eu^n(x,t_2))\right)\cdot \frac{\partial \varphi_i}{\partial x_j}(x)\,dx\right| \le \\
&\le d_2 \sum_{i,j,k,l} \left|\int_{\mathbb{R}^N} \frac{\partial}{\partial x_l}\,(u^n_k(x,t_1) - u^n_k(x,t_2))\cdot \frac{\partial \varphi_i}{\partial x_j}(x)\,dx\right| \le \\
&\le d_3 \sum_{i,k} \int_{\mathbb{R}^N} \left(|\xi|^{1-\epsilon}\,|\widehat{u^n_k}(\xi,t_1) - \widehat{u^n_k}(\xi,t_2)|\right)^2 \cdot \left(|\xi|^{1+\epsilon}\,|\widehat{\varphi}_i(\xi)|\right)^2 d\xi \le \\
&\le d_4\,\|u^n(\cdot,t_1) - u^n(\cdot,t_2)\|^2_{W^{1-\epsilon,2}} \cdot \|\varphi\|^2_{W^{1+\epsilon,2}} .
\end{aligned} \tag{40}$$

Let us remark that $\|u^n\|_{W^{\frac{\nu}{2}}((0,T);W^{1-\epsilon,2}(\Omega;\mathbb{R}^N))} \le d_5(T)$ by Lemma 1. We can proceed similarly in the case of the difference of the right-hand sides of the equation (30). It is sufficient to look at the term in (30) generated by the memory portion of the stress σ^M_{ij}.

It holds for $0 \le t_2 < t_1 < T$ that

$$\begin{aligned}
&\int_\Omega \left[\int_0^{t_1}\int_\Omega \frac{e_{ij}u^n(\xi,t_1) - e_{ij}u^n(\xi,\tau)}{|x-\xi|^\alpha}\,h(t_1-\tau)\,d\xi d\tau - \right.\\
&\left. - \int_0^{t_2}\int_\Omega \frac{e_{ij}u^n(\xi,t_2) - e_{ij}u^n(\xi,\sigma)}{|x-\xi|^\alpha}\,h(t_2-\sigma)\,d\xi d\sigma\right] e_{ij}\varphi(x)\,dx = \\
&= \int_\Omega\int_\Omega \left[\int_0^{t_1} (e_{ij}u^n(\xi,t_1) - e_{ij}u^n(\xi,t_1-s))\;h(s)\,ds - \right.\\
&\left. - \int_0^{t_2} (e_{ij}u^n(\xi,t_2) - e_{ij}u^n(\xi,t_2-s))\;h(s)\,ds\right] \frac{e_{ij}\varphi(x)}{|x-\xi|^\alpha}\,d\xi dx = \\
&= \int_{\mathbb{R}^N} \left\{\int_0^{t_2} (e_{ij}u^n(\xi,t_1) - e_{ij}u^n(\xi,t_2))\;h(s)\,ds - \right.\\
&- \int_0^{t_2} (e_{ij}u^n(\xi,t_1-s) - e_{ij}u^n(\xi,t_2-s))\;h(s)\,ds + \\
&\left. + \int_{t_2}^{t_1} (e_{ij}u^n(\xi,t_1) - e_{ij}u^n(\xi,t_1-s))\;h(s)\,ds\right\} \cdot \left(e_{ij}\varphi * \frac{1}{|\cdot|^\alpha}\right)(\xi)\,d\xi.
\end{aligned} \tag{41}$$

The last line of (30) contains three parts. We can write out the symmetric parts of the gradient and estimate corresponding integrals (similarly as in (40))

$$\begin{aligned}
&\left|\int_0^{t_2}\int_{\mathbb{R}^N} \left(\frac{\partial \varphi_i}{\partial x_j} * \frac{1}{|\cdot|^\alpha}\right)\left(\frac{\partial u^n_k}{\partial x_l}(t_1) - \frac{\partial u^n_k}{\partial x_l}(t_2)\right) d\xi\, h(s)\,ds\right| \le \\
&\le d_6(T)\,\|\varphi\|_{W^{1+\epsilon,2}} \cdot \|u^n(t_1) - u^n(t_2)\|_{W^{1-\epsilon,2}} \le \\
&\le d_7(T)\,\|u^n(t_1) - u^n(t_2)\|_{W^{1-\epsilon,2}}
\end{aligned} \tag{42}$$

The second integral can be estimated

$$(43)\quad \left|\int_0^{t_2}\int_{\mathbb{R}^N}\left(\frac{\partial\varphi_i}{\partial x_j}*\frac{1}{|\cdot|^\alpha}\right)\left(\frac{\partial u_k^n}{\partial x_l}(t_1-s)-\frac{\partial u_k^n}{\partial x_l}(t_2-s)\right)d\xi\, h(s)\, ds\right|\le$$
$$\le d_8(T)\,\|\varphi\|_{W^{1+\epsilon,2}}\cdot\int_0^{t_2}\|u^n(t_1-s)-u^n(t_2-s)\|_{W^{1-\epsilon,2}}h(s)\,ds\le$$
$$\le d_9(T)\left\{\int_0^{t_2}\|u^n(t_1)-u^n(t_2)\|^2_{W^{1-\epsilon,2}}ds\right\}^{\frac{1}{2}}$$

and

$$(44)\quad \int_0^T\int_0^{t_1}\int_0^{t_2}\frac{\|u^n(t_1-s)-u^n(t_2-s)\|^2_{W^{1-\epsilon,2}}}{|(t_1-s)-(t_2-s)|^{\nu+1}}\le d_{10}(T)\,\|u^n\|_{W^{\frac{\nu}{2}}((0,T);W^{1-\epsilon,2}(\Omega;\mathbb{R}^N))}\,.$$

Analogously we get

$$(45)\quad \int_{t_2}^{t_1}\int_{\mathbb{R}^N}\left(\frac{\partial\varphi_i}{\partial x_j}*\frac{1}{|\cdot|^\alpha}\right)\left(\frac{\partial u_k^n}{\partial x_l}(t_1)-\frac{\partial u_k^n}{\partial x_l}(t_1-s)\right)d\xi\, h(s)\,ds\le$$
$$\le d_{11}(T)\int_{t_2}^{t_1}\|u^n(t_1)-u^n(t_1-s)\|_{W^{1-\epsilon,2}}\,h(s)\,ds\le$$
$$\le d_{11}(T)\int_{t_2}^{t_1}\|u^n(t_1)-u^n(t_1-s)\|_{W^{1,2}}\,h(s)\,ds.$$

Hence

$$(46)\quad \int_0^T\int_0^{t_1}\left[\int_{t_2}^{t_1}\|u^n(t_1)-u^n(t_1-s)\|_{W^{1,2}}\,h(s)\,ds\right]^2\frac{dt_2dt_1}{(t_1-t_2)^{\nu+1}}\le$$
$$\le\sup_{\tau\in\langle 0,T\rangle}\|u^n(.,\tau)\|^2_{W^{1,2}}\int_0^T\int_0^{t_1}\left(\int_{t_2}^{t_1}h(s)\,ds\right)^2\frac{dt_2dt_1}{(t_1-t_2)^{\nu+1}}\le$$
$$\le d_{12}(T)\,\|u^n\|^2_{L^\infty((0,T);W^{1,2}(\Omega;\mathbb{R}^N))}\,.$$

Lemma 2 follows from definitions and estimates (42), (43), (46) and Lemma 1. ■

4. Interpolation.

Let $1<\mu<\frac{3}{2}$ and $-\frac{3}{2}<\beta<\frac{3}{2}$. We can introduce spaces $W^{\mu,2}((0,T);W_o^{\beta,2}(\Omega))$ by Definition 1. Then $v\in W^{\mu,2}((0,T);W_o^{\beta,2}(\Omega))$ may be expanded into the double Fourier series

$$(47)\quad v=\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}c_{i,j}\cdot h_i(t)\cdot v^j(x),$$

where

$$(48)\quad h_o(t)=\frac{1}{\sqrt{T}},\qquad h_i(t)=\sqrt{\frac{2}{T}}\cdot cos\frac{i\pi}{T}\cdot t,\quad i=1,2,\ldots.$$

We use the equivalent norm

$$\|v\|^2_{W^{\mu,2}((0,T);W^{\beta,2}_o(\Omega))} \approx \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} c_{i,j}^2 \cdot (1+i^2)^{\mu} \cdot \lambda_j^{\beta}. \tag{49}$$

Lemma 3. Let $0<\delta<\frac{1}{2}$, $0<\epsilon<\frac{1}{2}$, $0\le\gamma\le 1$. Then there exists a constant $C_3>0$ such that

$$\|v\|_{W^{(1+\delta)\cdot\gamma,2}((0,T);W^{-(1+\epsilon)\cdot\gamma,2}(\Omega))} \le C_3 \cdot \|v\|^{1-\gamma}_{L^2((0,T);L^2(\Omega))} \cdot \|v\|^{\gamma}_{W^{1+\delta,2}((0,T);W^{-1-\epsilon,2}(\Omega))}. \tag{50}$$

Proof: We compute directly that

$$\begin{aligned} \|v\|_{W^{(1+\delta)\cdot\gamma,2}((0,T);W^{-(1+\epsilon)\cdot\gamma,2}(\Omega))} &= \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} c_{ij}^2 \cdot \left(1+i^2\right)^{(1+\delta)\cdot\gamma} \cdot \lambda_j^{-(1+\epsilon)\cdot\gamma} = \\ &= \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} c_{ij}^{2\gamma} \cdot \left(1+i^2\right)^{\gamma\cdot(1+\delta)} \cdot \lambda_j^{-\gamma\cdot(1+\epsilon)} \cdot c_{ij}^{2(1-\gamma)} \le \\ &\le C_3 \cdot \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\left(c_{ij}^2 \cdot \left(1+i^2\right)^{(1+\delta)} \cdot \lambda_j^{-(1+\epsilon)}\right)^{\gamma} \cdot \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\left(c_{ij}^2 \cdot \left(1+i^2\right)^{0} \cdot \lambda_j^{0}\right)^{1-\gamma} = \\ &= C_3 \cdot \|v\|^{1-\gamma}_{L^2((0,T);L^2(\Omega))} \cdot \|v\|^{\gamma}_{W^{1+\delta,2}((0,T);W^{-1-\epsilon,2}(\Omega))}. \blacksquare \end{aligned} \tag{51}$$

5. Existence of a weak solution.

The following theorem establishes the Lipschitz continuity of the operator σ^M.

Theorem 1. There exists $p>0$, independent of T, such that

$$\|\sigma^M u - \sigma^M v\|_{L^2((0,T);L^2(\Omega;\mathbb{R}^{N^2}))} \le \lambda p \cdot \|eu - ev\|_{L^2((0,T);L^2(\Omega;\mathbb{R}^{N^2}))}. \tag{52}$$

Proof: We use twice Schwarz inequality

$$\begin{aligned} \frac{1}{\lambda^2}\left(\sigma_{ij}^M u - \sigma_{ij}^M v\right)^2 &= \\ &= \left(\int_0^t \int_\Omega \left[(e_{ij}u(\xi,t) - e_{ij}v(\xi,t)) - (e_{ij}u(\xi,\tau) - e_{ij}v(\xi,\tau))\right] \cdot \frac{h(t-\tau)}{|x-\xi|^\alpha}\, d\xi d\tau\right)^2 \le \\ &\le \int_\Omega (e_{ij}u(\xi,t) - e_{ij}v(\xi,t))^2 \frac{d\xi}{|x-\xi|^\alpha} \cdot \int_\Omega \frac{d\xi}{|x-\xi|^\alpha} \cdot \left(\int_0^t h(t-\tau)\, d\tau\right)^2 + \\ &\quad + \int_\Omega \int_0^t (e_{ij}u(\xi,\tau) - e_{ij}v(\xi,\tau))^2 \frac{h(t-\tau)}{|x-\xi|^\alpha} d\tau d\xi \cdot \int_\Omega \frac{d\xi}{|x-\xi|^\alpha} \cdot \int_0^t h(t-\tau)\, d\tau. \end{aligned} \tag{53}$$

We denote $p_1 = \int_0^\infty h(\tau)\, d\tau$ and $p_2 = \int_{B(0,diam\Omega)} \frac{1}{|x|^\alpha}\, dx$. Changing the order of the integration we go on to compute

(54)

$$\begin{aligned} \frac{1}{\lambda^2}\left\|\sigma^M u - \sigma^M v\right\|^2_{L^2(L^2)} &\le p_1^2 \cdot p_2 \cdot \int_\Omega \left(\int_0^T (e_{ij}u(\xi,t) - e_{ij}v(\xi,t))^2\, dt\right) \cdot \left(\int_\Omega \frac{d\xi)}{|x-\xi|^\alpha}\right) d\xi + \\ &+ p_1 \cdot p_2 \cdot \int_\Omega \int_0^T (e_{ij}u(\xi,\tau) - e_{ij}v(\xi,\tau))^2 \cdot \left(\int_\tau^T h(t-\tau)\, dt\right) \cdot \left(\int_\Omega \frac{d\xi)}{|x-\xi|^\alpha}\right) d\tau d\xi \le \\ &\le 2p_1^2 p_2^2 \|eu - ev\|^2_{L^2(L^2)}. \blacksquare \end{aligned}$$

Theorem 2 (existence of weak solutions). Let us consider the equation (4) — (6) with the boundary and the initial conditions (7) and (8). Let introduced assumptions be satisfied; moreover the following conditions hold:

$$\nu > N - \alpha \qquad (55)$$
$$c_0 \kappa > p\lambda$$

where c_o is the constant in Korn's inequality.

Then the problem (4) — (8) possesses weak solutions u on $(0,\infty)$. The solutions satisfy

$$\begin{aligned} (56) \qquad & u \in L^\infty((0,\infty); W_o^{1,2}(\Omega;\mathbb{R}^N)) \\ & \dot{u} \in L^\infty((0,\infty); L^2(\Omega;\mathbb{R}^N)) \\ & \ddot{u} \in L^2((0,T); W^{-1,2}(\Omega;\mathbb{R}^N)) \\ & u \in W^{\frac{\nu}{2},2}((0,T); W^{1-\frac{N-\alpha}{2},2}(\Omega;\mathbb{R}^N)) \\ & \ddot{u} \in W^{\frac{\nu}{2},2}((0,T); W^{-1-\frac{N-\alpha}{2},2}(\Omega;\mathbb{R}^N)) \end{aligned}$$

for all $T > 0$.

Proof: Let us choose any $T > 0$. As $\dot{u}_i^n$ is a bounded sequence in $L^2((0,T); L^2(\Omega))$ and $W^{1+\frac{\nu}{2},2}((0,T); W^{-1-\frac{N-\alpha}{2},2}(\Omega))$, we get from Lemma 3 (we put $v = \dot{u}_i^n$) that $\dot{u}_i^n$ is bounded also in the space $W^{\gamma(1+\frac{\nu}{2}),2}((0,T); W^{-\gamma(1+\frac{N-\alpha}{2}),2}(\Omega))$, where $\frac{1}{1+\frac{\nu}{2}} < \gamma < \frac{1}{1+\frac{N-\alpha}{2}}$. It is possible to choose such γ, because $\nu > N - \alpha$. The space $W^{\gamma(1+\frac{\nu}{2}),2}((0,T); W^{-\gamma(1+\frac{N-\alpha}{2}),2}(\Omega))$, is compactly embedded into the space $W^{1,2}((0,T); W^{-1,2}(\Omega))$. Thus we can choose a subsequence u^{n_k} which converges to a certain function u in the following sense:

$$\begin{aligned} (57) \qquad & u^{n_k} \rightharpoonup u && \text{in} \quad L^2((0,T); W_o^{1,2}(\Omega;\mathbb{R}^N)) \\ & \dot{u}^{n_k} \rightharpoonup \dot{u} && \text{in} \quad L^2((0,T); L^2(\Omega;\mathbb{R}^N)) \\ & u^{n_k} \rightharpoonup u && \text{in} \quad W^{\frac{\nu}{2},2}((0,T); W^{1-\frac{N-\alpha}{2},2}(\Omega;\mathbb{R}^N)) \\ & \ddot{u}^{n_k} \rightharpoonup \ddot{u} && \text{in} \quad W^{\frac{\nu}{2},2}((0,T); W^{-1-\frac{N-\alpha}{2},2}(\Omega;\mathbb{R}^N)) \\ & \ddot{u}^{n_k} \rightarrow \ddot{u} && \text{in} \quad L^2((0,T); W^{-1,2}(\Omega;\mathbb{R}^N)) \ . \end{aligned}$$

Now, let P_n be the projection operator from $L^2((0,T); W_o^{1,2}(\Omega;\mathbb{R}^N))$ to the space spanned by the vectors $c_j(t)\cdot w^j(x)$, where $c_j \in L^2((0,T))$, $j = 1,\dots,n$. We have

$$(58) \qquad P_n u \rightarrow u \qquad \text{v} \quad L^2((0,T); W_o^{1,2}(\Omega;\mathbb{R}^N)).$$

We put $v = u^n - P_n u$ as a test function into equality (30) obtaining

$$\begin{aligned} (59) \quad & \int_0^T \int_\Omega \ddot{u}_i^{n_k} \cdot (u_i^{n_k} - (P_{n_k}u)_i)\, dxdt + \int_0^T \int_\Omega \frac{\partial W}{\partial e_{ij}}(eu^{n_k}) \cdot e_{ij}\,(u^{n_k} - P_{n_k}u)\, dxdt - \\ & - \lambda \int_0^T \int_\Omega \left[\int_0^t \int_\Omega (e_{ij}u^{n_k}(\xi,\tau) - e_{ij}u^{n_k}(\xi,t)) \cdot \frac{d\xi}{|x-\xi|^\alpha} \cdot h(t-\tau)\, d\tau \right] \cdot e_{ij}\,(u^{n_k} - P_{n_k}u)\, dxdt = \\ & = \int_0^T \int_\Omega f_i \cdot (u_i^{n_k} - (P_{n_k}u)_i)\, dxdt. \end{aligned}$$

The first and the last integral tends to 0. We obtain a lower estimate from the condition of ellipticity (9) and Korn's inequality

$$\int_0^T \int_\Omega \frac{\partial W}{\partial e_{ij}} \left(e\left(u^{n_k} - P_{n_k}u\right)\right) \cdot e_{ij}\left(u^{n_k} - P_{n_k}u\right)\, dxdt \geq \tag{60}$$
$$\geq \kappa c_0 \left\|u^{n_k} - P_{n_k}u\right\|^2_{L^2((0,T);W^{1,2}_o(\Omega;\mathbb{R}^N))}.$$

As σ^M is Lipschitz continuous we have

$$\int_0^T \int_\Omega \sigma^M_{ij}\left(u^{n_k} - P_{n_k}u\right) \cdot e_{ij}\left(u^{n_k} - P_{n_k}u\right)\, dxdt \leq \tag{61}$$
$$\leq \lambda \cdot p \cdot \left\|u^{n_k} - P_{n_k}u\right\|^2_{L^2((0,T);W^{1,2}_o(\Omega;\mathbb{R}^N))}.$$

As

$$\int_0^T \int_\Omega \frac{\partial W}{\partial e_{ij}} \left(eP_{n_k}u\right) \cdot e_{ij}\left(u^{n_k} - P_{n_k}u\right)\, dxdt \to 0, \tag{62}$$
$$\int_0^T \int_\Omega \sigma^M_{ij}\left(P_{n_k}u\right) \cdot e_{ij}\left(u^{n_k} - P_{n_k}u\right)\, dxdt \to 0$$

and $\kappa c_0 - \lambda p > 0$, then $u^{n_k} - P_{n_k}u \to 0$ and also $u^{n_k} \to u$ in $L^2((0,T);W^{1,2}_o(\Omega;\mathbb{R}^N))$. The existence of the required weak solutions follows as a direct consequence of (30). ■

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František Mošna
Charles University
Faculty of Pedagogy
M.D.Rettigové 4
116 39 Praha 1
Czech Republic

Jindřich Nečas
Charles University
Mathematical Institute, Faculty of Mathematics and Physics
Sokolovská 83
186 00 Praha 8
Department of Mathematical Sciences
Nothern Illinois University
DeKalb
Illinois 60115